

# A FEFFERMAN-STEIN INEQUALITY FOR THE CARLESON OPERATOR

DAVID BELTRAN

**ABSTRACT.** We provide a Fefferman-Stein type weighted inequality for maximally modulated Calderón-Zygmund operators that satisfy *a priori* weak type unweighted estimates. This inequality corresponds to a maximally modulated version of a result of Pérez. Applying it to the Hilbert transform we obtain the corresponding Fefferman-Stein inequality for the Carleson operator  $\mathcal{C}$ , that is  $\mathcal{C} : L^p(M^{|p|+1}w) \rightarrow L^p(w)$  for any  $1 < p < \infty$  and any weight function  $w$ , with bound independent of  $w$ . We also provide a maximal-multiplier weighted theorem, a vector-valued extension, and more general two-weighted inequalities. Our proof builds on a recent work of Di Plinio and Lerner combined with some results on Orlicz spaces developed by Pérez.

## 1. INTRODUCTION

Let  $M$  denote the Hardy-Littlewood maximal operator. In 1971, Fefferman and Stein [13] proved that there is a constant  $C_n < \infty$ <sup>1</sup> such that for any  $1 < p < \infty$

$$(1) \quad \int_{\mathbb{R}^n} |Mf(x)|^p w(x) dx \leq C_n p' \int_{\mathbb{R}^n} |f(x)|^p Mw(x) dx$$

holds for all weight functions  $w$ , where  $p'$  denotes the conjugate exponent of  $p$ , that is  $1/p + 1/p' = 1$ . By weight we mean a non-negative locally integrable function.

Inequalities like (1) have been of considerable interest in recent years. On an informal level, given an operator  $U$  and an exponent  $p \in [1, \infty)$ , one studies if there is a constant  $C < \infty$  such that

$$(2) \quad \int_{\mathbb{R}^n} |Uf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \mathfrak{M}w(x) dx$$

holds for all admissible functions  $f$  and weights  $w$ , where  $\mathfrak{M}$  is a suitable maximal operator. By an elementary duality argument, one can use inequality (2) to transfer bounds from  $\mathfrak{M}$  to  $U$ , that is,

$$(3) \quad \|U\|_{L^q \rightarrow L^{\tilde{q}}} \lesssim \|\mathfrak{M}\|_{L^{(\tilde{q}/p)'} \rightarrow L^{(q/p)'}}^{1/p}$$

for  $q, \tilde{q} \geq p$ . Then, for any fixed exponent  $p$ , it is of particular interest to identify a maximal operator  $\mathfrak{M}$  satisfying (2) which is *optimal*, in the sense that Lebesgue space bounds for  $\mathfrak{M}$  allow one to obtain optimal Lebesgue space bounds for  $U$  via (3). See Bennett and Harrison [2], Bennett [1] and Córdoba and Rogers [6] for recent examples of such optimal control by maximal operators.

We consider the case of a Calderón-Zygmund operator on  $\mathbb{R}^n$ , that is, an  $L^2$  bounded operator represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad x \notin \text{supp } f,$$

where the kernel  $K$  satisfies

- (i)  $|K(x, y)| \leq \frac{C}{|x-y|^n}$  for all  $x \neq y$ ;
- (ii)  $|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x-x'|^\delta}{|x-y|^{n+\delta}}$  for some  $0 < \delta \leq 1$  when  $|x-x'| < |x-y|/2$ .

<sup>1</sup>Supported by ERC Grant 307617.

<sup>1</sup>Here and throughout the paper we use the letter  $C$  to denote a constant that may change from line to line that is, in particular, independent of the function  $f$  and the weight  $w$ . We also use the notation  $A \lesssim B$  to denote that there is a constant  $C$  such that  $A \leq CB$ .

Córdoba and Fefferman [5] showed that for  $s > 1$ ,  $1 < p < \infty$ , there is a constant  $C < \infty$  such that

$$(4) \quad \int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_s w(x) dx$$

holds for any weight  $w$ , where  $M_s w(x) = (Mw^s(x))^{1/s}$ <sup>2</sup>. Observe that for each  $s > 1$ , the operator  $M_s$  fails to be bounded on  $L^q$  for  $1 < q \leq s$ . Thus, for a fixed  $1 < p < \infty$ ,  $M_s w$  is not an optimal maximal operator, since via the inequality (3) we can only recover  $L^q$  bounds for  $T$  in the restricted range  $p \leq q < ps'$ , missing the exponents in  $[ps', \infty)$ ; we recall that  $T$  is an  $L^q$  bounded operator for  $1 < q < \infty$ . This problem was resolved by Wilson [33] in the range  $1 < p \leq 2$  and by Pérez [28] in the whole range  $1 < p < \infty$ , who showed that for  $1 < p < \infty$ , there is a constant  $C < \infty$  such that

$$(5) \quad \int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M^{|p|+1} w(x) dx$$

holds for any weight  $w$ . Here  $[p]$  denotes the integer part of  $p$  and  $M^{|p|+1}$  denotes the  $([p] + 1)$ -fold composition of  $M$ . The operator  $M^{|p|+1}$  is bounded on  $L^q$ ,  $1 < q < \infty$ , for any  $p$ . Thus, given  $1 < p < \infty$ , it is an optimal maximal operator since we can recover via (3) the  $L^q$  boundedness of  $T$  for the whole range  $p \leq q < \infty$ . Furthermore, their result is best possible in the sense that it fails if  $M^{|p|+1}$  is replaced by  $M^{|p|}$ . It should be noted that for each  $r > 1$  and  $k \geq 1$ , the pointwise estimate  $M^k w(x) \leq CM_r w(x)$  holds for some constant  $C$  independent of  $w$ .

The goal of this paper is to extend (5) to a broad class of maximally modulated Calderón-Zygmund operators studied previously by Grafakos, Martell and Soria [14] and Di Plinio and Lerner [10]. Let  $\Phi = \{\phi_\alpha\}_{\alpha \in A}$  be a family of real-valued measurable functions indexed by an arbitrary set  $A$ . Then the maximally modulated Calderón-Zygmund operator  $T^\Phi$  is defined by

$$(6) \quad T^\Phi f(x) = \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_\alpha} f)(x)|,$$

where  $\mathcal{M}^{\phi_\alpha} f(x) = e^{2\pi i \phi_\alpha(x)} f(x)$ . We will consider operators  $T^\Phi$  such that for some  $r_0 > 1$  satisfy the *a priori* weak-type inequalities

$$(7) \quad \|T^\Phi f\|_{L^{r,\infty}} \lesssim \psi(r) \|f\|_r$$

for  $1 < r \leq r_0$ , where  $\psi(r)$  is a function that captures the dependence of the operator norm on  $r$ . This definition is motivated by the Carleson operator,

$$\mathcal{C}f(x) = \sup_{\alpha \in \mathbb{R}} \left| \text{p.v.} \int_{\mathbb{R}} \frac{e^{2\pi i \alpha y}}{x - y} f(y) dy \right|$$

since it can be recovered from (6) by setting  $T = H$  and  $\Phi$  to be the family of functions given by  $\phi_\alpha(x) = \alpha x$  for  $\alpha \in \mathbb{R}$ . Expressing  $\mathcal{C}f$  in terms of  $\hat{f}$  allows it to be reconciled with the classical expression for the Carleson maximal operator in terms of partial Fourier integrals.

Implicit in the work of Di Plinio and Lerner [10] there is the following analogue of the estimate (4) for maximally modulated Calderón-Zygmund operators<sup>3</sup>.

**Theorem 1.1.** *Let  $T^\Phi$  be a maximally modulated Calderón-Zygmund operator satisfying (7). Then for  $1 < s < 2$  and  $1 < p < \infty$  there is a constant  $C < \infty$  such that for any weight  $w$*

$$(8) \quad \int_{\mathbb{R}^n} |T^\Phi f(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_s w(x) dx.$$

Note that the inequality (4) can be recovered from (8) simply by taking  $\phi_\alpha \equiv 0$  for all  $\alpha$ . As in the case of (4), for any fixed  $1 < p < \infty$  and  $1 < s < 2$ , Theorem 1.1 does not allow one to recover the full range of Lebesgue space bounds for  $T^\Phi$  from those for  $M_s$  via (3).

<sup>2</sup>This can also be seen as a consequence of the  $A_p$  theory, since  $M_s w \in A_1 \subset A_p$  for  $p > 1$ , with constant independent of  $w$ , and  $w(x) \leq M_s w(x)$ .

<sup>3</sup>Again, this result can be seen as a consequence of the  $A_\infty$  theory in [14]. In the case of the Carleson operator  $\mathcal{C}$  the result follows from the  $A_p$  theory in [18].

One can address this question and obtain optimal control of  $T^\Phi$  by combining the ideas developed by Pérez in [28] and [29] with Di Plinio and Lerner's argument [10]. The main result of this paper is the following.

**Theorem 1.2.** *Let  $T^\Phi$  be a maximally modulated Calderón-Zygmund operator satisfying (7). Then for any  $1 < p < \infty$  there is a constant  $C < \infty$  such that for any weight  $w$*

$$(9) \quad \int_{\mathbb{R}^n} |T^\Phi f(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M^{[p]+1} w(x) dx.$$

*This is best possible in the sense that  $[p] + 1$  cannot be replaced by  $[p]$ .*

Of course, one can recover the estimate (5) from Theorem 1.2. As observed for (5), given  $1 < p < \infty$ , the control given by the maximal operator  $M^{[p]+1}$  is optimal here.

Indeed Theorem 1.2 can be viewed as a corollary of a more precise statement, that allows one to replace  $M^{[p]+1}$  by a sharper class of maximal operators. This strategy is the same as the one in Pérez [28] for the case of unmodulated Calderón-Zygmund operators.

Let  $A$  be a Young function, that is,  $A : [0, \infty) \rightarrow [0, \infty)$  is a continuous, convex, increasing function with  $A(0) = 0$  and such that  $A(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . We say that a Young function  $A$  is doubling if there exists a positive constant  $C$  such that  $A(2t) \leq CA(t)$  for  $t > 0$ . For each cube  $Q \subset \mathbb{R}^n$ , we define the Luxemburg norm of  $f$  over  $Q$  by

$$\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$

and the maximal operator  $M_A$  by

$$M_A f(x) = \sup_{Q \ni x} \|f\|_{A,Q},$$

where  $f$  is a locally integrable function and the supremum is taken over all cubes  $Q$  in  $\mathbb{R}^n$  containing  $x$ .

In this context we are able to characterize the class of Young functions for which a Fefferman-Stein inequality holds with controlling maximal operator  $\mathfrak{M} = M_A$ .

**Theorem 1.3.** *Let  $T^\Phi$  be a maximally modulated Calderón-Zygmund operator satisfying (7). Suppose that  $A$  is a doubling Young function satisfying*

$$(10) \quad \int_c^\infty \left( \frac{t}{A(t)} \right)^{p'-1} \frac{dt}{t} < \infty$$

*for some  $c > 0$ . Then for any  $1 < p < \infty$  there is a constant  $C < \infty$  such that for any weight  $w$*

$$(11) \quad \int_{\mathbb{R}^n} |T^\Phi f(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_A w(x) dx.$$

In the unmodulated setting, Pérez [28] pointed out that condition (10) is necessary for (11) to hold for the Riesz transforms. Hence it becomes a necessary condition for Theorem 1.3 to be stated in such a generality, characterizing the class of Young functions for which (11) holds.

It is well known that the Carleson operator  $\mathcal{C}$  satisfies condition (7), so one can deduce the corresponding Fefferman-Stein weighted inequality.

**Corollary 1.4.** *Let  $\mathcal{C}$  be the Carleson operator. Then for any  $1 < p < \infty$  there is a constant  $C < \infty$  such that for every weight  $w$*

$$(12) \quad \int_{\mathbb{R}} |\mathcal{C}f(x)|^p w(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p M^{[p]+1} w(x) dx.$$

Weighted inequalities for the Carleson operator have been previously studied by many authors. Hunt and Young [18] established the  $L^p(w)$  boundedness of  $\mathcal{C}$  for  $1 < p < \infty$  and  $w \in A_p$ . Later Grafakos, Martell and Soria [14] gave new weighted inequalities for weights in  $A_\infty$ , as well as vector-valued inequalities for  $\mathcal{C}$ . More recently, Do and Lacey [11] gave weighted estimates for a variation norm version of  $\mathcal{C}$  in the

context of  $A_p$  theory that strengthened the results in [18]. Finally, Di Plinio and Lerner [10] obtained  $L^p(w)$  bounds for  $\mathcal{C}$  in terms of the  $[w]_{A_q}$  constants for  $1 \leq q \leq p$ . Note that inequality (12) does not fall within the scope of the classical  $A_p$  theory.

This paper is organised as follows. In Section 2 we present some powerful results due to Lerner [24, 23, 10] that allow one to bound in norm  $T^\Phi$  by the so-called dyadic sparse operators. In Section 3 we present the results obtained by Pérez in [28] and [29] concerning the maximal operator associated to a Young function. In Section 4 we give the proof of Theorem 1.3 and how to apply it to deduce Theorem 1.2. Section 5 contains some applications that can be deduced from our main result and in Section 6 we present a vector-valued extension of the main theorem. Finally, Section 7 is devoted to extending our result to more general two-weighted inequalities.

**Acknowledgements.** The author would like to thank his supervisor Jon Bennett for his continuous support and for many valuable comments on the exposition of this paper.

## 2. A NORM ESTIMATE BY DYADIC SPARSE OPERATORS

Here we present a result in [10] that allows one to reduce the proof of (11) to a Fefferman-Stein inequality for dyadic sparse operators. This reduction rests on a certain local mean oscillation estimate. Such estimates have been developed by Lerner and other authors and have become a powerful technique over the last few years. See, for instance, [21, 22, 23, 10, 16, 19, 20]. We have considered it instructive to recall this local mean oscillation estimate approach since it will be used as well for the vector-valued extension presented in Section 6.

Let  $\mathcal{D}$  be a general dyadic grid, that is a collection of cubes such that

- (i) any  $Q \in \mathcal{D}$  has sidelength  $2^k$ ,  $k \in \mathbf{Z}$ ;
- (ii) for any  $Q, R \in \mathcal{D}$ , we have  $Q \cap R \in \{Q, R, \emptyset\}$ ;
- (iii) the cubes of a fixed sidelength  $2^k$  form a partition of  $\mathbb{R}^n$ .

Given a cube  $Q_0$  we denote by  $\mathcal{D}(Q_0)$  the set of all dyadic cubes with respect to  $Q_0$ , that is, the cubes obtained by dividing dyadically  $Q_0$  and its descendants into  $2^n$  subcubes.

We say that  $\mathcal{S}$  is a *sparse* family of cubes if for any cube  $Q \in \mathcal{S}$  there is a measurable subset  $E(Q) \subset Q$  such that  $|Q| \leq 2|E(Q)|$  and the sets  $\{E(Q)\}_{Q \in \mathcal{S}}$  are pairwise disjoint.

Given a measurable function  $f$  and a cube  $Q$ , the local mean oscillation of  $f$  on  $Q$  is defined by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda|Q|)$$

for  $0 < \lambda < 1$ , where  $f^*$  denotes the non-increasing rearrangement of  $f$ .

The median value of  $f$  over a cube  $Q$ , denoted by  $m_f(Q)$ , is a nonunique real number such that

$$|\{x \in Q : f(x) > m_f(Q)\}| \leq |Q|/2$$

and

$$|\{x \in Q : f(x) < m_f(Q)\}| \leq |Q|/2.$$

Given a measurable function  $f$  and a cube  $Q_0$ , one can control pointwise the value of  $f$  on  $Q_0$  in terms of the median value of  $f$  on  $Q_0$  and the local mean oscillation of  $f$  in a sparse family of cubes. A first version of this result was obtained by Lerner [21]; see [19] for the following refined version.

**Theorem 2.1** ([19]). *Let  $f$  be a measurable function on  $\mathbb{R}^n$  and  $Q_0$  be a fixed cube. Then there exists a sparse family of cubes  $\mathcal{S} \subset \mathcal{D}(Q_0)$  such that*

$$|f(x) - m_f(Q_0)| \leq 2 \sum_{Q \in \mathcal{S}} \omega_{\frac{1}{2^{n+2}}}(f; Q) \chi_Q(x)$$

for a.e.  $x \in Q_0$ .

Di Plinio and Lerner [10] applied the above local mean oscillation estimate to  $T^\Phi f$  to obtain an estimate for  $\|T^\Phi f\|_{L^p(w)}$  in terms of the  $L^p(w)$  bounds of certain dyadic sparse operators, that come up naturally from the local mean oscillation of  $T^\Phi f$  on a cube  $Q$ .

**Proposition 2.2** ([10]). *Let  $T^\Phi$  be a maximally modulated Calderón-Zygmund operator satisfying (7). Then, for any cube  $Q \subset \mathbb{R}^d$  and any  $1 < r \leq r_0$ ,*

$$(13) \quad \omega_\lambda(T^\Phi f; Q) \lesssim \psi(r) \left( \frac{1}{|Q|} \int_Q |f|^r \right)^{1/r} + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f| \right),$$

where  $\bar{Q} = 2\sqrt{n}Q$ .

Given a sparse family  $\mathcal{S}$  consider the dyadic sparse operator

$$\mathcal{A}_{r,\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q |f|^r \right)^{1/r} \chi_Q(x).$$

Then one has the following norm estimate result.

**Proposition 2.3** ([10]). *Let  $T^\Phi$  be a maximally modulated Calderón-Zygmund operator satisfying (7). Let  $1 < p < \infty$  and let  $w$  be an arbitrary weight. Then*

$$\|T^\Phi f\|_{L^p(w)} \lesssim \inf_{1 < r \leq r_0} \left\{ \psi(r) \sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{r,\mathcal{S}} f\|_{L^p(w)} \right\}$$

where the supremum is taken over all dyadic grids  $\mathcal{D}$  and all sparse families  $\mathcal{S} \subset \mathcal{D}$ .

### 3. BOUNDS FOR THE MAXIMAL OPERATOR

Let  $B$  be a Young function. We define the complementary Young function  $\bar{B}$  to be the Legendre transform of  $B$ , that is

$$\bar{B}(t) = \sup_{s>0} \{st - B(s)\}, \quad t > 0.$$

We have that  $\bar{B}$  is also a Young function, and it satisfies

$$t \leq B^{-1}(t) \bar{B}^{-1}(t) \leq 2t$$

for  $t > 0$ . For all functions  $f, g$  and all cubes  $Q \subset \mathbb{R}^n$ , the following generalised Hölder's inequality holds,

$$\frac{1}{|Q|} \int_Q f(x)g(x)dx \leq \|f\|_{B,Q} \|g\|_{\bar{B},Q}.$$

Pérez [29] characterised the Young functions  $B$  such that  $M_B$  is of strong type  $(p, p)$  for  $p > 1$  and established that the  $L^p$  boundedness is equivalent to certain weighted inequalities for  $M_B$  and related maximal operators.

**Theorem 3.1** ([29]). *Let  $1 < p < \infty$ . Let  $A$  and  $B$  be doubling Young functions satisfying  $\bar{B}(t) = A(t^{p'})$ . Then the following are equivalent:*

(i)  *$B$  satisfies the  $B_p$  condition, denoted by  $B \in B_p$ : there is a constant  $c > 0$  such that*

$$\int_c^\infty \frac{B(t)}{t^p} \frac{dt}{t} \approx \int_c^\infty \left( \frac{t^{p'}}{\bar{B}(t)} \right)^{p-1} \frac{dt}{t} < \infty.$$

(ii) *There is a constant  $c > 0$  such that*

$$\int_c^\infty \left( \frac{t}{A(t)} \right)^{p-1} \frac{dt}{t} < \infty.$$

(iii) There is a constant  $C < \infty$  such that

$$\int_{\mathbb{R}^n} M_B f(x)^p dx \leq C \int_{\mathbb{R}^n} f(x)^p dx$$

for all non-negative functions  $f$ .

(iv) There is a constant  $C < \infty$  such that

$$\int_{\mathbb{R}^n} M_B f(x)^p u(x) dx \leq C \int_{\mathbb{R}^n} f(x)^p M u(x) dx$$

for all non-negative functions  $f$  and any weight  $u$ .

(v) There is a constant  $C < \infty$  such that

$$(14) \quad \int_{\mathbb{R}^n} M f(x)^p \frac{u(x)}{(M_A w(x))^{p-1}} dx \leq C \int_{\mathbb{R}^n} f(x)^p \frac{M u(x)}{w(x)^{p-1}} dx$$

for all non-negative functions  $f$  and any weights  $u, w$ .

A classical result from Coifman and Rochberg [4] asserts that for any locally integrable function  $w$  such that  $Mw(x) < \infty$  a.e. and  $0 < \delta < 1$ , the function  $(Mw)^\delta(x)$  is an  $A_1$  weight with constant independent of  $w$ . More precisely,

$$M((Mw)^\delta)(x) \leq C_n \frac{1}{1-\delta} (Mw)^\delta(x)$$

for almost all  $x \in \mathbb{R}^n$ . As Pérez [28] remarks, this result still holds when one replaces the Hardy-Littlewood maximal function by the maximal operator  $M_A$ .

**Proposition 3.2** ([8], Proposition 5.32). *Let  $A$  be a Young function. If  $0 < \delta < 1$ , then  $(M_A w)^\delta \in A_1$  with  $A_1$  constant independent of  $w$ . In particular,*

$$M((M_A w)^\delta)(x) \leq C_n \frac{1}{1-\delta} (M_A w)^\delta(x)$$

for almost all  $x \in \mathbb{R}^n$ .

#### 4. PROOF OF THEOREM 1.3

In this section we give a proof of Theorem 1.3 and we use it, thanks to an observation due to Pérez [28, 29], to deduce Theorem 1.2. Our proof follows a similar pattern of a proof of Di Plinio and Lerner in [10].

As seen in Section 2, the boundedness of  $T^\Phi$  can be essentially reduced to the uniform boundedness of the dyadic sparse operators  $\mathcal{A}_{r,S}$ . In particular, we have the following Fefferman-Stein inequality for  $\mathcal{A}_{r,S}$ .

**Theorem 4.1.** *Let  $\mathcal{D}$  be a dyadic grid and  $\mathcal{S} \subset \mathcal{D}$  a sparse family of cubes. Suppose that  $A$  is a Young function satisfying (10). Then for  $1 < p < \infty$ , there is a constant  $C_{n,p,A} < \infty$  independent of  $\mathcal{S}$ ,  $\mathcal{D}$  and the weight  $w$  such that*

$$\|\mathcal{A}_{r,S} f\|_{L^p(w)} \leq C_{n,p,A} \left( \left( \frac{p+1}{2r} \right)' \right)^{1/r} \|f\|_{L^p(M_A w)}$$

holds for any  $1 < r < \frac{p+1}{2}$  and any non-negative function  $f$ .

*Proof.* We first linearize the operator  $\mathcal{A}_{r,S}$ . For any  $Q$ , by  $L^p$  duality, there exists  $g_Q$  supported in  $\bar{Q}$  such that  $\frac{1}{|\bar{Q}|} \int_{\bar{Q}} g_Q^{r'} = 1$  and

$$\left( \frac{1}{|\bar{Q}|} \int_{\bar{Q}} f^r \right)^{1/r} = \frac{1}{|\bar{Q}|} \int_{\bar{Q}} f g_Q.$$

We can thus define a linear operator  $L$  by

$$Lh(x) = \sum_{Q \in \mathcal{S}} \left( \frac{1}{|\bar{Q}|} \int_{\bar{Q}} h g_Q \right) \chi_Q(x).$$

Note that  $L(f) = \mathcal{A}_{r,\mathcal{S}}f$ . Then, in order to obtain an estimate for  $\|\mathcal{A}_{r,\mathcal{S}}\|_{L^p(w)}$  independent of  $\mathcal{S}$  and  $\mathcal{D}$ , it is enough to obtain the corresponding estimate for  $\|Lh\|_{L^p(w)}$  uniformly in the functions  $g_Q$ . By duality, the estimate

$$\|Lh\|_{L^p(w)} \leq C_{n,p,A} \left( \left( \frac{p+1}{2r} \right)' \right)^{1/r} \|h\|_{L^p(M_A w)}$$

is equivalent to

$$(15) \quad \|L^*h\|_{L^{p'}((M_A w)^{1-p'})} \leq C_{n,p,A} \left( \left( \frac{p+1}{2r} \right)' \right)^{1/r} \|h\|_{L^{p'}(w^{1-p'})}$$

where  $L^*$  denotes the  $L^2(\mathbb{R}^n)$ -adjoint operator of  $L$ . Since  $A$  satisfies (10), one can apply Theorem 3.1 with  $p$  replaced by  $p'$ . Then using (14) with  $u \equiv 1$ , the estimate (15) follows from

$$(16) \quad \|L^*h\|_{L^{p'}((M_A w)^{1-p'})} \leq C_n \left( \left( \frac{p+1}{2r} \right)' \right)^{1/r} \|Mh\|_{L^{p'}((M_A w)^{1-p'})}.$$

We focus then on obtaining (16). By duality, there exists  $\eta \geq 0$  such that  $\|\eta\|_{L^p(M_A w)} = 1$  and

$$\|L^*h\|_{L^{p'}((M_A w)^{1-p'})} = \int_{\mathbb{R}^n} L^*(h)\eta dx = \int_{\mathbb{R}^n} hL\eta dx.$$

By Hölder's inequality and the  $L^{r'}$  boundedness of  $g_Q$ ,

$$(17) \quad \begin{aligned} \int_{\mathbb{R}^n} hL\eta dx &= \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q \eta g_Q \right) \int_Q h \leq \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q \eta^r \right)^{1/r} \int_Q h \\ &\leq \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q \eta^r \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right) (2\sqrt{n})^n |Q| \\ &= (2\sqrt{n})^n \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q \eta^r \left( \frac{1}{|Q|} \int_Q h \right)^{\frac{r}{p+1}} \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right)^{\frac{p}{p+1}} |Q|. \end{aligned}$$

Recall that by definition of the Hardy-Littlewood maximal operator

$$(18) \quad \frac{1}{|Q|} \int_Q h(x) dx \leq Mh(y)$$

holds for every  $y \in \bar{Q}$ . Combining this and the sparseness of  $\mathcal{S}$

$$(19) \quad \begin{aligned} (17) &\leq 2(2\sqrt{n})^n \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q \left( (Mh)^{\frac{1}{p+1}} \eta \right)^r \right)^{1/r} \left( \frac{1}{|Q|} \int_Q h \right)^{\frac{p}{p+1}} |E(Q)| \\ &\leq 2(2\sqrt{n})^n \sum_{Q \in \mathcal{S}} \int_{E(Q)} M_r((Mh)^{\frac{1}{p+1}} \eta) (Mh)^{\frac{p}{p+1}} dx \\ &\leq 2(2\sqrt{n})^n \int_{\mathbb{R}^n} M_r((Mh)^{\frac{1}{p+1}} \eta) (Mh)^{\frac{p}{p+1}} dx, \end{aligned}$$

where we have used that  $(E(Q))_{Q \in \mathcal{S}}$  are pairwise disjoint and that (18) also holds for  $y \in E(Q) \subset Q \subset \bar{Q}$ . By Hölder's inequality with exponents  $\rho = \frac{p+1}{2}$  and  $\rho' = \frac{p+1}{p-1}$ ,

$$(20) \quad \begin{aligned} (19) &= 2(2\sqrt{n})^n \int_{\mathbb{R}^n} M_r((Mh)^{\frac{1}{p+1}} \eta) (M_A w)^{\frac{1}{p+1}} (Mh)^{\frac{p}{p+1}} (M_A w)^{-\frac{1}{p+1}} dx \\ &\leq 2(2\sqrt{n})^n \|M_r((Mh)^{\frac{1}{p+1}} \eta)\|_{L^{\frac{p+1}{2}}((M_A w)^{1/2})} \|Mh\|_{L^{p'}((M_A w)^{1-p'})}^{\frac{p}{p+1}}. \end{aligned}$$

For  $r < \frac{p+1}{2}$ , we can apply the classical Fefferman-Stein inequality (1) to the first term in (20)

$$\|M_r((Mh)^{\frac{1}{p+1}} \eta)\|_{L^{\frac{p+1}{2}}((M_A w)^{1/2})} \leq C_n \left( \left( \frac{p+1}{2r} \right)' \right)^{1/r} \|(Mh)^{\frac{1}{p+1}} \eta\|_{L^{\frac{p+1}{2}}(M(M_A w)^{1/2})},$$

and by Proposition 3.2

$$\|(Mh)^{\frac{1}{p+1}} \eta\|_{L^{\frac{p+1}{2}}(M(M_A w)^{1/2})} \leq C_n \|(Mh)^{\frac{1}{p+1}} \eta\|_{L^{\frac{p+1}{2}}((M_A w)^{1/2})}.$$

Finally, by an application of Hölder's inequality with  $\rho = 2p'$  and  $\rho' = \frac{2p}{p+1}$

$$\begin{aligned} \|(Mh)^{\frac{1}{p+1}}\eta\|_{L^{\frac{p+1}{2}}((M_A w)^{1/2})} &= \left( \int_{\mathbb{R}^n} \left( (Mh)^{\frac{1}{2}} (M_A w)^{-\frac{1}{2p}} \right) \left( \eta^{\frac{p+1}{2}} (M_A w)^{\frac{p+1}{2p}} \right) dx \right)^{\frac{2}{p+1}} \\ &\leq \|Mh\|_{L^{p'}((M_A w)^{1-p'})}^{\frac{1}{p+1}} \|\eta\|_{L^p(M_A w)} = \|Mh\|_{L^{p'}((M_A w)^{1-p'})}^{\frac{1}{p+1}}, \end{aligned}$$

where the last equality holds since  $\|\eta\|_{L^p(M_A w)} = 1$ . So altogether,

$$\|L^*h\|_{L^{p'}((M_A w)^{1-p'})} \leq 2(2\sqrt{n})^n C_n \left( \left( \frac{p+1}{2r} \right)' \right)^{1/r} \|Mh\|_{L^{p'}((M_A w)^{1-p'})}.$$

This concludes the proof.  $\square$

We are now able to prove Theorem 1.3.

*Proof of Theorem 1.3.* By Proposition 2.3, it is enough to show that for any  $1 < p < \infty$ ,

$$\inf_{1 < r \leq r_0} \left\{ \psi(r) \sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{r, \mathcal{S}} f\|_{L^p(w)} \right\} \lesssim \|f\|_{L^p(M_A w)}.$$

By Theorem 4.1,

$$\sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{r, \mathcal{S}} f\|_{L^p(w)} \leq C_{n, p, A} \left( \left( \frac{p+1}{2r} \right)' \right)^{1/r} \|f\|_{L^p(M_A w)}$$

for any  $1 < r < \frac{p+1}{2}$ , since the bound was independent of  $\mathcal{D}, \mathcal{S}$ .

For every  $p > 1$ , consider

$$r_p = \min \left\{ r_0, 1 + \frac{p-1}{3} \right\} = \min \left\{ r_0, \frac{p+2}{3} \right\}.$$

We have that  $1 < r_p \leq r_0$  and  $r_p < \frac{p+1}{2}$ . Then

$$\|T^\Phi f\|_{L^p(w)} \lesssim \psi(r_p) \sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{r_p, \mathcal{S}} f\|_{L^p(w)} \leq \psi(r_p) C_{n, p, A} \left( \left( \frac{p+1}{2r_p} \right)' \right)^{1/r_p} \|f\|_{L^p(M_A w)}.$$

This concludes the proof.  $\square$

Observe that this proof of Theorem 1.3 could be extended to other operators whose bounds depend in a suitable way on those of  $\mathcal{A}_{r, \mathcal{S}}$ . This will be the case of the vector-valued extension presented in Section 6.

Theorem 1.2 may be deduced from Theorem 1.3. Given the Young function  $A(t) = t \log^{[p]}(1+t)$ , which clearly satisfies (10), there exists a constant  $C < \infty$  such that  $M_A w(x) \leq C M^{[p]+1} w(x)$  for any weight  $w$ . This observation is due to Pérez [28, 29].

Finally, one cannot replace  $[p] + 1$  by  $[p]$  in the statement of Theorem 1.2, since the resulting inequality is shown to be false for the (unmodulated) Hilbert transform [28].

## 5. APPLICATIONS

**5.1. Maximal multiplier of bounded variation.** The essence of the classical Marcinkiewicz multiplier theorem is the observation that a multiplier of bounded variation on the line often satisfies the same norm inequalities as the Hilbert transform. In particular, if  $m$  is a bounded variation multiplier and  $T_m$  is its associated operator, one can deduce

$$\int_{\mathbb{R}} |T_m f(x)|^p w(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p M^{[p]+1} w(x) dx$$



for any weight  $w$ . Using Corollary 1.4, one can deduce the analogous maximal-multiplier inequality in the sense of Oberlin [27]. Consider the maximal-multiplier operator

$$\mathcal{M}_{BV}f(x) = \sup_{m: \|m\|_{BV} \leq 1} |(m\hat{f})^\sim(x)|$$

where the supremum is taken over all functions whose variation norm is less or equal than 1. Recall that the variation norm is defined by

$$(21) \quad \|m\|_{BV} := \|m\|_\infty + \sup_{N, \xi_0 < \dots < \xi_N} \left( \sum_{i=1}^N |f(\xi_i) - f(\xi_{i-1})| \right),$$

where the supremum is over all strictly increasing finite length sequences of real numbers. The second term in the right hand side of (21) is the total variation of  $m$ .

**Theorem 5.1.** *For  $1 < p < \infty$ , there is a constant  $C < \infty$  such that*

$$(22) \quad \int_{\mathbb{R}} |\mathcal{M}_{BV}f(x)|^p w(x) dx \leq C \int_{\mathbb{R}} |f(x)|^p M^{[p]+1} w(x) dx$$

holds for any weight  $w$ .

*Proof.* Since  $m$  is of global bounded variation,

$$T_m f(x) = cf(x) + \int_{\mathbb{R}} S_{(t, \infty)} f(x) dm(t) \leq cf(x) + \int_{\mathbb{R}} \mathcal{C}f(x) dm(t),$$

where  $dm$  denotes the Lebesgue-Stieltjes measure associated to  $m$ . Then

$$\sup_{m: \|m\|_{BV} \leq 1} |T_m f(x)| \leq c|f(x)| + |\mathcal{C}f(x)| \sup_{m: \|m\|_{BV} \leq 1} \int_{\mathbb{R}} |dm|(t) \leq c|f(x)| + |\mathcal{C}f(x)|,$$

where the last inequality follows since the integral of  $|dm|$  corresponds to the total variation of  $m$ . The proof concludes by taking  $L^p(w)$  norms and using Corollary 1.4.  $\square$

**Remark 5.2.** Let  $m$  be a multiplier of bounded variation and let  $m_t(\xi) = m(t\xi)$ . Consider the maximal operator associated to these multipliers, that is,

$$T_m^* f(x) = \sup_{t>0} |(m_t \hat{f})^\sim(x)|.$$

Since  $m$  and  $m_t$  have the same variation norm, we have  $T_m^* f(x) \leq \|m\|_{BV} \mathcal{M}_{BV} f(x)$ , so the inequality (22) also holds for  $T_m^*$  in place of  $\mathcal{M}_{BV}$ .

**5.2. Carleson-type operators in higher dimensions.** Concerning higher dimensional Carleson operators, the Fefferman-Stein weighted inequality also holds for the operator

$$\mathcal{C}_P f(x) := \sup_{t>0} \left| \int_{tP} \hat{f}(\xi) e^{ix \cdot \xi} d\xi \right|,$$

where  $P$  is a polyhedron with finitely many faces and the origin in its interior. Indeed, Fefferman deduced in [12] that the norm of this operator is bounded by the norm of the one-dimensional Carleson operator  $\mathcal{C}$  in any Banach space.

**5.3. The Polynomial Carleson operator.** Let  $d \in \mathbb{N}$ . The polynomial Carleson operator is defined as

$$(23) \quad \mathcal{C}_d f(x) := \sup_{\deg(P) \leq d} \left| \text{p. v.} \int_{\mathbb{R}} \frac{e^{iP(y)}}{y} f(x-y) dy \right|,$$

where the supremum is taken over all real-coefficient polynomials  $P$  of degree at most  $d$ . Note that for  $d = 1$  one recovers the definition of the Carleson operator.

It was conjectured by Stein that the operator  $\mathcal{C}_d$  is bounded in  $L^p$  for  $1 < p < \infty$ . In the case of periodic functions, this conjecture has been recently solved by Lie [25] via time-frequency analysis techniques; see [26] for his previous work for  $\mathcal{C}_2$ .

One may write  $\mathcal{C}_d f(x) = \sup_{P \in \mathcal{P}} |H^\mathbb{T}(\mathcal{M}^P f)(x)|$  for  $x \in \mathbb{T}$ , where  $\mathcal{M}^P f(x) = e^{iP(x)} f(x)$  and  $H^\mathbb{T}$  denotes the periodic Hilbert transform. Straightforward modifications in the proof of Theorem 1.2 yield a similar result for the periodic case and thus, for any  $1 < p < \infty$  there is a constant  $C < \infty$  such that for any weight  $w$

$$\int_{\mathbb{T}} |\mathcal{C}_d f(x)|^p w(x) dx \leq C \int_{\mathbb{T}} |f(x)|^p M^{\lfloor p \rfloor + 1} w(x) dx.$$

## 6. VECTOR-VALUED EXTENSIONS

In this section we provide a vector-valued extension of Theorem 1.3. As we already mentioned in Section 2, we prove such a result via the local mean oscillation estimate approach.

Let  $T^\Phi$  be a maximally modulated Calderón-Zygmund operator. Given a sequence of functions  $f = (f_j)_{j \in \mathbb{N}}$ , consider the vector-valued extension of  $T^\Phi$ , given by  $\bar{T}^\Phi f = (T^\Phi f_j)_{j \in \mathbb{N}}$ . For  $q \geq 1$ , we define the function  $|f|_q$  by

$$|f(x)|_q = \left( \sum_{j=1}^{\infty} |f_j(x)|^q \right)^{1/q}.$$

As in the case of  $T^\Phi$ , we will assume that the operator  $\bar{T}^\Phi$  satisfies the a priori weak type inequalities

$$(24) \quad \|\bar{T}^\Phi f\|_{L^{r,\infty}(\ell^q)} \lesssim \psi(r) \|f\|_{L^r(\ell^q)}$$

for  $1 < r \leq r_0$  and some  $r_0 > 1$ . Theorem 1.3 extends naturally for  $\bar{T}^\Phi$  in  $L^p(\ell^q)$ .

**Theorem 6.1.** *For  $q \geq 1$ , let  $\bar{T}^\Phi$  be a vector-valued maximally modulated Calderón-Zygmund operator satisfying (24) and  $1 < p < \infty$ . Suppose that  $A$  is a doubling Young function satisfying*

$$\int_c^\infty \left( \frac{t}{A(t)} \right)^{p'-1} \frac{dt}{t} < \infty$$

for some  $c > 0$ . Then there is a constant  $C < \infty$  such that for any weight  $w$

$$\int_{\mathbb{R}^n} |\bar{T}^\Phi f(x)|_q^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|_q^p M_A w(x) dx.$$

To prove Theorem 6.1, we apply Proposition 2.1 to  $|\bar{T}^\Phi f|_q$ . To this end, we need to obtain a bound for the local mean oscillation of  $|\bar{T}^\Phi f|_q$  on a cube  $Q$ .

**Proposition 6.2.** *Let  $q \geq 1$  and  $\bar{T}^\Phi$  be a vector-valued maximally modulated Calderón-Zygmund operator satisfying (24). Then, for any  $1 < r \leq r_0$ ,*

$$(25) \quad \omega_\lambda(|\bar{T}^\Phi f|_q; Q) \lesssim \psi(r) \left( \frac{1}{|Q|} \int_Q |f|_q^r \right)^{1/r} + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f|_q \right).$$

Note that the bound obtained in (25) is the same as the one in (13) with  $f$  replaced by  $|f|_q$ . By the weak-type estimate (24), one can see that  $m_{|\bar{T}^\Phi f|_q}(Q) \rightarrow 0$  as  $|Q| \rightarrow \infty$ . Thus, one can obtain an analogue of Proposition 2.3 with  $T^\Phi f$  replaced by  $|\bar{T}^\Phi f|_q$  and  $f$  replaced by  $|f|_q$ . This combined with Theorem 4.1 gives Theorem 6.1.

We proceed now to the proof of Proposition 6.2. The ideas used are quite standard; see [30] for a similar argument in the case of vector-valued Calderón-Zygmund operators or [10] for the similar proof of Proposition 2.2.

*Proof of Proposition 6.2.* Write  $f = f^0 + f^\infty$ , where  $f^0 = f \chi_{\bar{Q}}$ . Denote by  $c_Q$  the centre of the cube  $Q$ . Then

$$\left| |\bar{T}^\Phi f(x)|_q - |\bar{T}^\Phi f^\infty(c_Q)|_q \right| \leq |\bar{T}^\Phi f(x) - \bar{T}^\Phi f^\infty(c_Q)|_q$$

$$\begin{aligned}
&= \left( \sum_{j=1}^{\infty} \left| \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_{\alpha}} f_j)(x)| - \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(c_Q)| \right|^q \right)^{1/q} \\
&\leq \left( \sum_{j=1}^{\infty} \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_{\alpha}} f_j)(x) - T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(c_Q)|^q \right)^{1/q} \\
&\leq |\bar{T}^{\Phi} f^0(x)|_q + \left( \sum_{j=1}^{\infty} \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(x) - T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(c_Q)|^q \right)^{1/q}.
\end{aligned}$$

Since  $\bar{T}^{\Phi}$  is of weak-type  $(r, r)$ , it is straightforward to see that

$$(|\bar{T}^{\Phi} f^0|_q \chi_Q)^*(\lambda |Q|) \lesssim \psi(r) \left( \frac{1}{|Q|} \int_{\bar{Q}} |f|_q^r \right)^{1/r}.$$

For the second term, if  $x \in Q$ ,

$$\begin{aligned}
&\left( \sum_{j=1}^{\infty} \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(x) - T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(c_Q)|^q \right)^{1/q} \\
&= \left( \sum_{j=1}^{\infty} \sup_{\alpha \in A} \left| \int_{\mathbb{R}^n \setminus \bar{Q}} [K(x, z) - K(c_Q, z)] \mathcal{M}^{\phi_{\alpha}} f_j^{\infty}(z) dz \right|^q \right)^{1/q} \\
&\leq \left( \sum_{j=1}^{\infty} \left( \int_{\mathbb{R}^n \setminus \bar{Q}} |K(x, z) - K(c_Q, z)| |f_j^{\infty}(z)| dz \right)^q \right)^{1/q} \\
&\leq \int_{\mathbb{R}^n \setminus 2Q} \left( \sum_{j=1}^{\infty} |K(x, z) - K(c_Q, z)|^q |f_j^{\infty}(z)|^q \right)^{1/q} dz \\
&\leq \int_{\mathbb{R}^n \setminus 2Q} \left( \sum_{j=1}^{\infty} \frac{|x - c_Q|^{q\delta}}{|x - z|^{qn+q\delta}} |f_j^{\infty}(z)|^q \right)^{1/q} dz \\
&= \sum_{m=1}^{\infty} \int_{2^{m+1}Q \setminus 2^m Q} \left( \sum_{j=1}^{\infty} \frac{|x - c_Q|^{q\delta}}{|x - z|^{qn+q\delta}} |f_j^{\infty}(z)|^q \right)^{1/q} dz \\
&\leq \sum_{m=1}^{\infty} \frac{\ell(Q)^{\delta}}{(2^m \ell(Q))^{n+\delta}} \int_{2^{m+1}Q \setminus 2^m Q} \left( \sum_{j=1}^{\infty} |f_j^{\infty}(z)|^q \right)^{1/q} dz \\
&\lesssim \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f|_q \right),
\end{aligned}$$

where we have used Minkowski integral inequality and the regularity of the Calderón-Zygmund kernel  $K$ . Then, choosing  $c = |\bar{T}^{\Phi} f^0(c_Q)|_q$  in the definition of  $\omega_{\lambda}(|\bar{T}^{\Phi} f|_q; Q)$ ,

$$\begin{aligned}
\omega_{\lambda}(|\bar{T}^{\Phi} f|_q; Q) &\leq (|\bar{T}^{\Phi} f^0|_q \chi_Q)^*(\lambda |Q|) \\
&\quad + \sup_{x \in Q} \left| \left( \sum_{j=1}^{\infty} \sup_{\alpha \in A} |T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(x) - T(\mathcal{M}^{\phi_{\alpha}} f_j^{\infty})(c_Q)|^q \right)^{1/q} \right| \\
&\leq \psi(r) \left( \frac{1}{|Q|} \int_{\bar{Q}} |f|_q^r \right)^{1/r} + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f|_q \right).
\end{aligned}$$

□

For  $q > 1$ , the vector-valued version of the Carleson operator is bounded in  $L^p$  for  $p > 1$  (see [31, 14]). Thus, for any  $1 < p, q < \infty$ , one may apply Theorem 6.1 and obtain the weighted inequality

$$\int_{\mathbb{R}} \left( \sum_{j=1}^{\infty} |\mathcal{C} f_j(x)|^q \right)^{p/q} w(x) dx \leq C \int_{\mathbb{R}} \left( \sum_{j=1}^{\infty} |f_j(x)|^q \right)^{p/q} M^{|p|+1} w(x) dx,$$

with  $C$  independent of the weight function  $w$ . The same result follows for the vector-valued version of  $\mathcal{C}_d$ , due to its boundedness in  $L^p$  for  $1 < p < \infty$  - see Remarks in [25].

## 7. TWO-WEIGHTED INEQUALITIES

Our approach to obtain weighted Fefferman-Stein inequalities can be extended to more general two-weighted inequalities. In the spirit of the work done for the Hardy-Littlewood maximal operator [29] and for Calderón-Zygmund operators [7, 21, 23], one can obtain sufficient conditions on a pair of weights  $(u, v)$  in order to  $T^\Phi : L^p(v) \rightarrow L^p(u)$ . With similar arguments to those of Theorem 1.3, one can prove that if, for some  $1 < r < \min\{r_0, p\}$ , a pair of weights  $(u, v)$  satisfy

$$[u, v]_{A, B} = \sup_{Q \subset \mathbb{R}^n} \|u^{1/p}\|_{A, Q} \|v^{-r/p}\|_{B, Q}^{1/r} < \infty,$$

where  $A$  and  $B$  are doubling Young functions such that  $\bar{A} \in B_{p'}$  and  $\bar{B} \in B_{\frac{p+1}{2r}}$ , there exists a constant  $C = C_{n, p, A, B, u, v} < \infty$  such that

$$\int_{\mathbb{R}^n} |T^\Phi f(x)|^p u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx.$$

However, there is an alternative way of obtaining such two-weighted inequalities that does not involve the linearisation and the adjoint operator argument in the proof given for Theorem 1.3. This approach follows the ideas of a similar two-weighted inequality for Calderón-Zygmund operators proved by Lerner in [23].

**Theorem 7.1.** *Let  $T^\Phi$  be a maximally modulated Calderón-Zygmund operator satisfying (7). Let  $1 < p < \infty$  and  $A, B$  be doubling Young functions such that  $\bar{A} \in B_{p'}$  and  $\bar{B} \in B_{p/r}$  for some  $1 < r < \min\{r_0, p\}$ . Let  $u$  and  $v$  be positive weights such that*

$$\sup_{Q \subset \mathbb{R}^n} \|u^{1/p}\|_{A, Q} \|v^{-r/p}\|_{B, Q}^{1/r} < \infty.$$

*Then there is a constant  $C = C_{n, p, A, B, u, v} < \infty$  such that*

$$(26) \quad \int_{\mathbb{R}^n} |T^\Phi f(x)|^p u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx.$$

*Proof.* By Proposition 2.3 it is enough to see that

$$\|\mathcal{A}_{r, \mathcal{S}} f\|_{L^p(u)} \leq C \|f\|_{L^p(v)}$$

with constant independent of the dyadic sparse family  $\mathcal{S}$ . By duality there exists  $g \in L^{p'}$ ,  $\|g\|_{p'} = 1$  such that

$$\left( \int_{\mathbb{R}^n} \mathcal{A}_{r, \mathcal{S}} f(x)^p u(x) dx \right)^{1/p} = \int_{\mathbb{R}^n} \mathcal{A}_{r, \mathcal{S}} f(x) u(x)^{1/p} g(x) dx.$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{A}_{r, \mathcal{S}} f(x) u(x)^{1/p} g(x) dx &= \sum_{Q \in \mathcal{S}} \left( \frac{1}{|\bar{Q}|} \int_{\bar{Q}} |f|^r \right)^{1/r} \int_Q u(x)^{1/p} g(x) dx \\ &= \sum_{Q \in \mathcal{S}} \left( \frac{1}{|\bar{Q}|} \int_{\bar{Q}} |f|^r v^{r/p} v^{-r/p} \right)^{1/r} \left( \frac{1}{|\bar{Q}|} \int_{\bar{Q}} u(x)^{1/p} g(x) dx \right) |\bar{Q}| \\ &\lesssim \sum_{Q \in \mathcal{S}} \|f^r v^{r/p}\|_{\bar{B}, \bar{Q}}^{1/r} \|v^{-r/p}\|_{B, \bar{Q}}^{1/r} \|u^{1/p}\|_{A, \bar{Q}} \|g\|_{\bar{A}, \bar{Q}} |E(Q)| \\ &\leq \sum_{Q \in \mathcal{S}} \int_{E(Q)} M_{\bar{B}}(f^r v^{r/p})(x)^{1/r} M_{\bar{A}} g(x) dx \\ &\leq \int_{\mathbb{R}^n} M_{\bar{B}}(f^r v^{r/p})(x)^{1/r} M_{\bar{A}} g(x) dx \\ &\leq \|M_{\bar{B}}(f^r v^{r/p})\|_{p/r}^{1/r} \|M_{\bar{A}} g\|_{p'} \\ &\lesssim \|f^r v^{r/p}\|_{p/r}^{1/r} \|g\|_{p'} \\ &= \|f\|_{L^p(v)}, \end{aligned}$$

where we have used Hölder's inequality for Young functions, the sparseness of the family  $\mathcal{S}$  and the boundedness of the operators  $M_{\bar{A}}$  and  $M_{\bar{B}}$ .  $\square$

**Remark 7.2.** The obvious vector-valued extensions considered in Section 6 also hold for this more general two-weighted case.

**Remark 7.3.** One may recover the Fefferman-Stein weighted inequalities (11) from Theorem 7.1 by considering the pair of weights  $(w, M_{\Gamma}w)$ , where  $\Gamma(t) = A(t^{1/p})$  and the Young function  $B(t) = t^{(p/r)'+\varepsilon}$ , that satisfies  $\bar{B} \in B_{p/r}$ . In this case, the constant  $C$  in (26) does not depend on  $w$ , since

$$\begin{aligned} [w, M_{\Gamma}w]_{A,B} &= \sup_{Q \subset \mathbb{R}^n} \|w^{1/p}\|_{A,Q} \|(M_{\Gamma}w)^{-r/p}\|_{B,Q}^{1/r} \\ &= \sup_{Q \subset \mathbb{R}^n} \|w\|_{\Gamma,Q}^{1/p} \left( \frac{1}{|Q|} \int_Q (M_{\Gamma}w)^{-(r/p)((p/r)'+\varepsilon)} \right)^{\frac{1}{(p/r)'+\varepsilon} \frac{1}{r}} \\ &\leq \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q (M_{\Gamma}w)^{-(r/p)((p/r)'+\varepsilon)} (M_{\Gamma}w)^{(1/p)((p/r)'+\varepsilon)r} \right)^{\frac{1}{(p/r)'+\varepsilon} \frac{1}{r}} \\ &= \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q 1 \right)^{\frac{1}{(p/r)'+\varepsilon} \frac{1}{r}} \\ &= 1, \end{aligned}$$

where the second equality follows from the definition of Luxemburg norm.

An advantage of this alternative proof for inequality (26) is that it can easily be adapted to a multilinear setting, since it does not involve any linear duality. In particular, one may obtain two-weighted inequalities for the multilinear Calderón-Zygmund operators introduced by Grafakos and Torres [15], that is, a multilinear operator  $T$  bounded from  $L^{q_1} \times \dots \times L^{q_m} \rightarrow L^q$  for some  $1 \leq q_1, \dots, q_m < \infty$  satisfying  $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$  and that can be represented as

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

for all  $x \notin \bigcap_{j=1}^m \text{supp } f_j$ , where the kernel  $K : (\mathbb{R}^n)^{m+1} \setminus \Delta \rightarrow \mathbb{R}$ , with  $\Delta = \{(x, y_1, \dots, y_m) : x = y_1 = \dots = y_m\}$ , satisfies the following size condition

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}},$$

and the regularity condition

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_m)| \leq \frac{A|y_j - y'_j|^\delta}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\delta}}$$

for some  $\delta > 0$  and all  $0 \leq j \leq m$ , whenever  $|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$ .

Via a local mean oscillation estimate, Damián, Lerner and Pérez [9] reduced norm estimates for such a multilinear operator  $T$  to norm estimates for a multilinear version of the dyadic sparse operator  $\mathcal{A}_{r,S}$ . Then, the proof of Theorem 7.1 can be adapted to the multilinear Calderón-Zygmund framework; given  $1 < p, p_1, \dots, p_m < \infty$  satisfying  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ , and  $(u, v_1, \dots, v_m)$  weights such that

$$\sup_{Q \subset \mathbb{R}^n} \|u^{1/p}\|_{A,Q} \prod_{i=1}^m \|v_i^{-1/p_i}\|_{B_i,Q} < \infty,$$

where  $A, B_1, \dots, B_m$  are doubling Young functions such that  $\bar{A} \in B_{p'}$  and  $\bar{B}_i \in B_{p_i}$ ,  $i = 1, \dots, m$ , one has

$$\int_{\mathbb{R}^n} |T(f_1, \dots, f_m)(x)|^p u(x) dx \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^n} |f_j(x)|^{p_j} v_j(x) dx \right)^{p/p_j}.$$

Of course the multilinear analogue of Remark 7.3 allows one to deduce a Fefferman-Stein weighted inequality for  $T$ , that is, given  $1 < p, p_1, \dots, p_m < \infty$  satisfying  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$  and a doubling Young

function  $A$  satisfying (10),

$$\int_{\mathbb{R}^n} |T(f_1, \dots, f_m)(x)|^p w(x) dx \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^n} |f_j(x)|^{p_j} M_A w(x) dx \right)^{p/p_j}.$$

This allows one to recover the result obtained by Hu [17] via different methods; Hu obtained the above inequality by induction on the level of linearity and using the linear result (5).

## 8. FURTHER REMARKS

**8.1. Lacunary Carleson operator.** Let  $\Lambda = \{\lambda_j\}_j$  be a lacunary sequence of integers, that is,  $\lambda_{j+1} \geq \theta \lambda_j$  for all  $j$  and for some  $\theta > 1$  and consider the lacunary Carleson maximal operator

$$\mathcal{C}_\Lambda f(x) = \sup_{j \in \mathbb{N}} \left| \text{p. v.} \int_{\mathbb{R}} \frac{e^{2\pi i \lambda_j y}}{x - y} f(y) dy \right|.$$

Of course one has the pointwise estimate  $\mathcal{C}_\Lambda f(x) \leq \mathcal{C}f(x)$ , so the Fefferman-Stein inequality (9) trivially holds for  $\mathcal{C}_\Lambda$ . This can be reconciled with a Fefferman-Stein inequality for  $\mathcal{C}_\Lambda$  obtained by more classical techniques. Consider the more classical version of the lacunary Carleson operator in terms of the lacunary partial Fourier integrals. Following the lines of [3],

$$S_\Lambda^* f(x) = \sup_k |S_{\lambda_k} f(x)| \leq c M f(x) + \left( \sum_k |S_{\lambda_k} f(f * \psi_k)(x)|^2 \right)^{1/2},$$

where  $\psi$  is a suitable Schwartz function and  $\widehat{S_{\lambda_k} f}(\xi) = \chi_{[-\lambda_k, \lambda_k]}(\xi) \widehat{f}(\xi)$ . Since  $S_{\lambda_k}$  satisfies the same Lebesgue space inequalities as the Hilbert transform, from the estimate (5) and weighted Littlewood-Paley theory (which can be obtained via a standard Rademacher function argument and the results from Pérez [28] and Wilson [34]), one can deduce the inequality (9) for  $\mathcal{C}_\Lambda$  with a higher number of compositions of the Hardy-Littlewood maximal operator  $M$ .

**8.2. Walsh-Carleson operator.** Following the lines of [10], one can obtain the corresponding Fefferman-Stein inequality for the Walsh-Carleson maximal operator  $\mathcal{W}$ . This operator is defined as

$$\mathcal{W}f(x) = \sup_{n \in \mathbb{N}} |\mathcal{W}_n f(x)|,$$

for  $x \in \mathbb{T} = [0, 1]$ , where  $\mathcal{W}_n$  denotes the  $n$ -th partial Walsh-Fourier sum, often considered as a discrete model of the Fourier case. We refer to [32] for definitions and elementary results on Walsh-Fourier series. It is proven in [10] that for  $1 < p < \infty$  and any weight  $w$ ,

$$\|\mathcal{W}f\|_{L^p(w)} \lesssim \inf_{1 < r \leq 2} \left\{ r' \sup_{\mathcal{S}} \|\tilde{\mathcal{A}}_{r, \mathcal{S}}\|_{L^p(w)} \right\},$$

where

$$\tilde{\mathcal{A}}_{r, \mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q |f|^r \right)^{1/r} \chi_Q(x)$$

and  $\mathcal{S} \subset \mathcal{D}(\mathbb{T})$  is a sparse family of dyadic cubes. Thus,

$$\int_{\mathbb{T}} |\mathcal{W}f(x)|^p w(x) dx \leq C \int_{\mathbb{T}} |f(x)|^p M^{\lfloor p \rfloor + 1} w(x) dx$$

follows by adapting the proof of Theorem 4.1 to the operators  $\tilde{\mathcal{A}}_{r, \mathcal{S}}$  and to functions defined in  $\mathbb{T}$ .

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SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, WATSON BUILDING, EDGBASTON, BIRMINGHAM, B15 2TT, UNITED KINGDOM

*E-mail address:* d.beltran@pgr.bham.ac.uk